



The Open
University

MST124

Essential mathematics 1

Exercise Booklet 4

1 Right-angled triangles

Exercise 1

- (a) Convert the following angles from degrees to radians.
- (i) 36° (ii) 150° (iii) 450°
- (b) Convert the following angles from radians to degrees.
- (i) $\frac{7\pi}{9}$ (ii) $\frac{2\pi}{3}$ (iii) $\frac{13\pi}{12}$

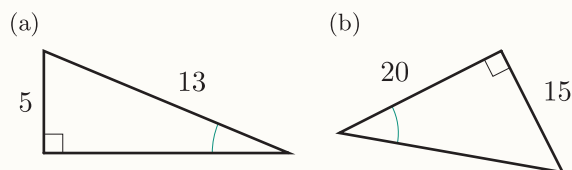
Exercise 2

Express the answer to each of the following question parts in terms of π .

- (a) Find the arc length of the following sectors of a circle.
- (i) Radius 5 cm, angle of sector $\frac{\pi}{6}$ radians.
- (ii) Radius 10 m, angle of sector $\frac{2\pi}{3}$ radians.
- (iii) Radius 8 cm, angle of sector 30° .
- (b) Find the perimeter of each sector in part (a).
- (c) Find the area of each sector in part (a).

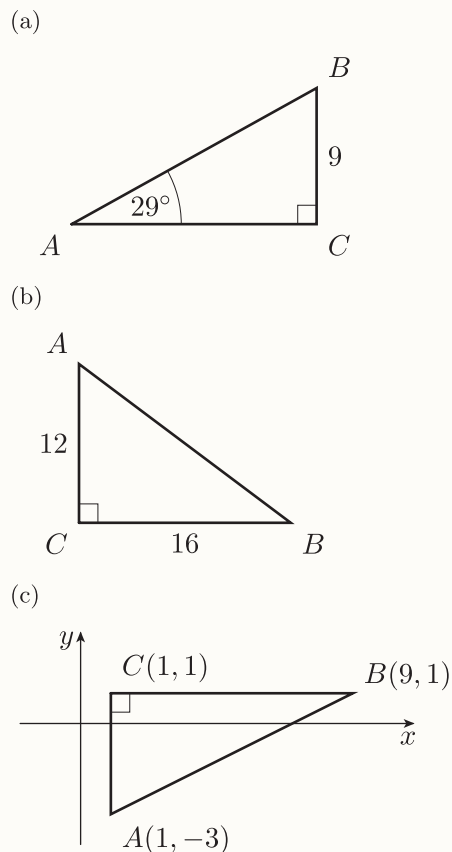
Exercise 3

For each of the triangles below, use Pythagoras' theorem to calculate the exact length of the third side, and hence find exact values for the sine, cosine and tangent of each of the marked angles.



Exercise 4

Find the unmarked angles and side lengths for each of the following three right-angled triangles. Give your answers to three significant figures (where they are not exact).



Exercise 5

Use the identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta},$$

$$\sin \theta = \cos \left(\frac{\pi}{2} - \theta \right) \text{ and}$$

$$\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$$

to show that

$$\tan \left(\frac{\pi}{2} - \theta \right) = \frac{1}{\tan \theta}.$$

2 Trigonometric functions

Exercise 6

- (a) For each of the following angles (measured in radians), determine in which quadrant it lies.
- (i) $-\frac{\pi}{4}$ (ii) $-\frac{7\pi}{6}$ (iii) $\frac{17\pi}{8}$
- (iv) $\frac{23\pi}{6}$
- (b) Illustrate each of your answers to part (a) by showing the location of the point P on the unit circle.
- (c) For each angle in part (a), write down an angle in the interval $(-\pi, \pi]$ that differs from the given angle by an integer multiple of 2π , ensuring that your answers agree with your diagrams in part (b) above.

Exercise 7

Find the values, where they are defined, of the following, without using a calculator.

- (a) $\sin\left(-\frac{3\pi}{2}\right)$, $\cos\left(-\frac{3\pi}{2}\right)$, $\tan\left(-\frac{3\pi}{2}\right)$
- (b) $\sin(-3\pi)$, $\cos(-3\pi)$, $\tan(-3\pi)$

Exercise 8

For each of the following angles, sketch the associated point P on the unit circle and find its coordinates. Hence find the sine, cosine and tangent of the angle.

- (a) $-\frac{\pi}{6}$ (b) $\frac{5\pi}{4}$

Exercise 9

Use the ASTC diagram and the special angles table to find the sine, cosine and tangent of each of the following angles.

- (a) $-\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{4\pi}{3}$

Exercise 10

Use the graphs of sine, cosine and tangent, and the special angles table, to confirm your answers to Exercise 9.

Exercise 11

Find all the solutions between -360° and 360° of the following equations. Give your answers both in degrees and radians.

- (a) $\sin \theta = \frac{1}{2}$ (b) $\cos \theta = -\frac{\sqrt{3}}{2}$

Exercise 12

Use the graph of the tangent function to find all the solutions between $-\pi$ and π of the equation $\tan \theta = 5$, in radians to two decimal places.

Exercise 13

- (a) Find all the solutions in the interval $[-2\pi, 2\pi]$ of the equation $\cos \theta = 0$.
- (b) Find all the solutions in the interval $[-\pi, 3\pi]$ of the equation $\sin \theta = 1$.

3 Sine and cosine rules

Exercise 14

A triangle ABC has $b = 9$, $A = 24^\circ$ and $C = 53^\circ$ (where the sides and angles are labelled in accordance with Figure 44 in Subsection 3.1 of Unit 4).

Find the other angle and the other side lengths, giving the side lengths to one decimal place.

Exercise 15

A triangle ABC has $A = 75.4^\circ$, $a = 13.8$ and $b = 6.5$ (where the sides and angles are labelled in accordance with Figure 44 in Subsection 3.1 of Unit 4).

Use the sine rule to calculate the angle C , in degrees to two decimal places.

Exercise 16

- (a) Given the information that a triangle ABC has $b = 14$, $c = 7$ and $C = 20^\circ$ (where the sides and angles are labelled in accordance with Figure 44 in Subsection 3.1 of Unit 4), find the possible values of the angle B , giving your answers to one decimal place.
- (b) Given the further piece of information that the longest side of the triangle is BC , find the other side length and other angles of the triangle.

Exercise 17

In this question give your answers to four decimal places.

A triangle ABC has $a = 4.6$, $b = 5.4$ and $C = 108^\circ$ (where the sides and angles are labelled in accordance with Figure 44 in Subsection 3.1 of Unit 4).

- (a) Use the cosine rule to find c .
- (b) Use the sine rule to find the other two angles.

Exercise 18

A triangle ABC has $a = 6$, $b = 14$ and $c = 11$ (where the sides and angles are labelled in accordance with Figure 44 in Subsection 3.1 of Unit 4).

Use the cosine rule to find the three angles of the triangle, to two decimal places.

Exercise 19

By using the formula $\text{area} = \frac{1}{2}ab \sin C$ for the area of a triangle ABC , calculate the area of a regular hexagon with sides of length 8 cm. Give your answer to two decimal places.

Exercise 20

Without using a calculator, find the equation of the straight line through the point $(5, 0)$ that makes an angle of $\frac{3\pi}{4}$ with the positive x -axis.

4 Further trigonometric identities

Exercise 21

Show that

$$\cos \theta = \sqrt{1 - \sin^2 \theta}$$

if $\cos \theta$ is positive and

$$\cos \theta = -\sqrt{1 - \sin^2 \theta}$$

if $\cos \theta$ is negative.

Exercise 22

Where they are defined, and without using a calculator, find exact values for each of the following.

(a) $\operatorname{cosec} \left(\frac{3\pi}{2} \right), \sec \left(\frac{3\pi}{2} \right), \cot \left(\frac{3\pi}{2} \right)$

(b) $\operatorname{cosec} \pi, \sec \pi, \cot \pi$

(c) $\operatorname{cosec} \left(-\frac{\pi}{2} \right), \sec \left(-\frac{\pi}{2} \right), \cot \left(-\frac{\pi}{2} \right)$

Exercise 23

Deduce each of the following trigonometric identities from an angle sum or angle difference identity.

(a) $\sin(x - \pi) = -\sin x$

(b) $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

(c) $\cos(x + \pi) = -\cos x$

(d) $\tan(x + \pi) = \tan x$

Exercise 24

Using the angle sum identities for sine, cosine and tangent, and without using a calculator or a graph, find the exact values of the following.

(a) $\sin\left(\frac{5\pi}{12}\right)$, $\cos\left(\frac{5\pi}{12}\right)$ and $\tan\left(\frac{5\pi}{12}\right)$

(b) $\sin\left(\frac{5\pi}{4}\right)$, $\cos\left(\frac{5\pi}{4}\right)$ and $\tan\left(\frac{5\pi}{4}\right)$

Exercise 25

Using your answers to Exercise 24, and without using a calculator, find the exact values of the following.

(a) $\sin\left(\frac{5\pi}{8}\right)$ (b) $\cos\left(\frac{5\pi}{8}\right)$

(c) $\sin\left(\frac{5\pi}{24}\right)$ (d) $\cos\left(\frac{5\pi}{24}\right)$

Solutions to exercises**Solution to Exercise 1**

(a) Applying the degrees to radians conversion formula gives the following.

(i) $36^\circ = \left(\frac{\pi}{180} \times 36\right)$ radians
 $= \frac{\pi}{5}$ radians

(ii) $150^\circ = \left(\frac{\pi}{180} \times 150\right)$ radians
 $= \frac{5\pi}{6}$ radians

(iii) $450^\circ = \left(\frac{\pi}{180} \times 450\right)$ radians
 $= \frac{5\pi}{2}$ radians

(b) Applying the radians to degrees conversion formula gives the following.

(i) $\frac{7\pi}{9}$ radians $= \left(\frac{180}{\pi} \times \frac{7\pi}{9}\right)^\circ = 140^\circ$

(ii) $\frac{2\pi}{3}$ radians $= \left(\frac{180}{\pi} \times \frac{2\pi}{3}\right)^\circ = 120^\circ$

(iii) $\frac{13\pi}{12}$ radians $= \left(\frac{180}{\pi} \times \frac{13\pi}{12}\right)^\circ = 195^\circ$

Solution to Exercise 2

(a) The arc length is given by the formula

$$\text{arc length} = r\theta,$$

where r is the radius of the circle and θ is the angle, measured in radians, subtended by the arc.

(i) Arc length $= 5 \times \frac{\pi}{6} = \frac{5\pi}{6}$ cm.

(ii) Arc length $= 10 \times \frac{2\pi}{3} = \frac{20\pi}{3}$ m.

(iii) $30^\circ = \frac{\pi}{180} \times 30 = \frac{\pi}{6}$ radians.
 Arc length $= 8 \times \frac{\pi}{6} = \frac{4\pi}{3}$ cm.

Exercise Booklet 4

- (b) The perimeter is the sum of the arc length and two times the radius.

$$\begin{aligned} \text{(i) Perimeter} &= \left(\frac{5\pi}{6} + 10 \right) \text{ cm} \\ &= \frac{5(\pi + 12)}{6} \text{ cm.} \end{aligned}$$

$$\begin{aligned} \text{(ii) Perimeter} &= \left(\frac{20\pi}{3} + 20 \right) \text{ m} \\ &= \frac{20(\pi + 3)}{3} \text{ m.} \end{aligned}$$

$$\begin{aligned} \text{(iii) Perimeter} &= \left(\frac{4\pi}{3} + 16 \right) \text{ cm} \\ &= \frac{4(\pi + 12)}{3} \text{ cm.} \end{aligned}$$

- (c) The area is given by the formula

$$\text{area} = \frac{1}{2}r^2\theta,$$

where r is the radius of the circle and θ is the angle of the sector measured in radians.

$$\text{(i) Area} = \frac{1}{2} \times 25 \times \frac{\pi}{6} = \left(\frac{25\pi}{12} \right) \text{ cm}^2.$$

$$\text{(ii) Area} = \frac{1}{2} \times 100 \times \frac{2\pi}{3} = \left(\frac{100\pi}{3} \right) \text{ m}^2.$$

$$\text{(iii) Area} = \frac{1}{2} \times 64 \times \frac{\pi}{6} = \left(\frac{16\pi}{3} \right) \text{ cm}^2.$$

Solution to Exercise 3

- (a) In this triangle the hypotenuse is 13, the opposite side is 5 and the adjacent side is unknown.

Using Pythagoras' theorem gives

$$5^2 + (\text{unknown side})^2 = 13^2.$$

So

$$(\text{unknown side})^2 = 169 - 25 = 144 = 12^2.$$

So the unknown side has length 12.

Call the marked angle θ . Then

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{5}{13}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{12}{13}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{5}{12}.$$

- (b) In this triangle the opposite side is 15 and the adjacent side is 20.

Using Pythagoras' theorem gives

$$15^2 + 20^2 = \text{hypotenuse}^2.$$

So

$$\text{hypotenuse}^2 = 225 + 400 = 625 = 25^2,$$

so the hypotenuse has length 25.

Call the marked angle θ . Then

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{15}{25} = \frac{3}{5}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{20}{25} = \frac{4}{5}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{15}{20} = \frac{3}{4}.$$

Solution to Exercise 4

(Note that the symbol $\angle$ means 'angle'. For example, $\angle A$ means the angle at vertex A .)

- (a) Since $BC = 9$ and $\angle A = 29^\circ$, we have $\angle B = 90^\circ - 29^\circ = 61^\circ$ and

$$AC = 9 \tan 61^\circ = 16.2 \text{ (to 3 s.f.)}$$

$$AB = \frac{9}{\sin 29^\circ} = 18.6 \text{ (to 3 s.f.)}.$$

- (b) By Pythagoras' Theorem, we have

$$AB^2 = 16^2 + 12^2 = 400,$$

so

$$AB = \sqrt{400} = 20.$$

The sine of $\angle B$ is

$$\frac{12}{20} = \frac{3}{5}.$$

Applying the inverse sine function on a calculator to this value gives the result

$$\angle B = 36.9^\circ \text{ (to 3 s.f.)}.$$

Hence the remaining angle is given by

$$\angle A = 90^\circ - \angle B = 53.1^\circ \text{ (to 3 s.f.)}.$$

- (c) We have

$$AC = 1 - (-3) = 4$$

and

$$BC = 9 - 1 = 8.$$

Hence, by Pythagoras' theorem,

$$AB^2 = 4^2 + 8^2 = 16 + 64 = 80,$$

so

$$AB = \sqrt{80} = 8.94 \text{ (to 3 s.f.)}.$$

The tangent of $\angle A$ is

$$\frac{8}{4} = 2.$$

Applying the inverse tangent function on a calculator to this value gives the result

$$\angle A = 63.4^\circ \text{ (to 3 s.f.)}.$$

Hence the remaining angle is given by

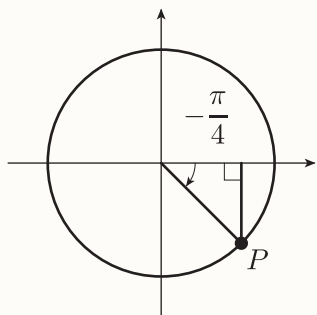
$$\angle B = 90^\circ - \angle A = 26.6^\circ \text{ (to 3 s.f.)}.$$

Solution to Exercise 5

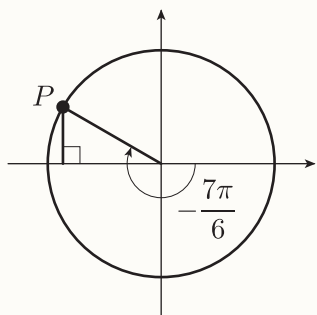
$$\begin{aligned} \tan\left(\frac{\pi}{2} - \theta\right) &= \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} \\ &= \frac{\cos \theta}{\sin \theta} \\ &= \frac{1}{\tan \theta} \end{aligned}$$

Solution to Exercise 6

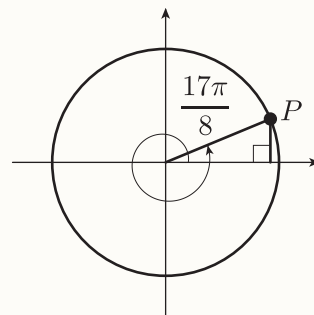
- (a) (i) Fourth quadrant
 (ii) Second quadrant
 (iii) First quadrant
 (iv) Fourth quadrant
- (b) (i)



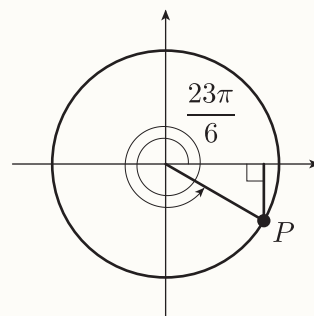
(ii)



(iii)



(iv)



- (c) (i) $-\frac{\pi}{4}$ differs from itself by an integer multiple of 2π .
 (ii) $\frac{5\pi}{6}$ differs from $-\frac{7\pi}{6}$ by an integer multiple of 2π .
 (iii) $\frac{\pi}{8}$ differs from $\frac{17\pi}{8}$ by an integer multiple of 2π .
 (iv) $-\frac{\pi}{6}$ differs from $\frac{23\pi}{6}$ by an integer multiple of 2π .

Solution to Exercise 7

- (a) $-\frac{3\pi}{2}$ differs from $\frac{\pi}{2}$ by an integer multiple of 2π . So

$$\sin\left(-\frac{3\pi}{2}\right) = \sin\frac{\pi}{2} = 1,$$

$$\cos\left(-\frac{3\pi}{2}\right) = \cos\frac{\pi}{2} = 0,$$

$$\tan\left(-\frac{3\pi}{2}\right) = \tan\frac{\pi}{2} \text{ is not defined.}$$

- (b) -3π differs from π by an integer multiple of 2π . So

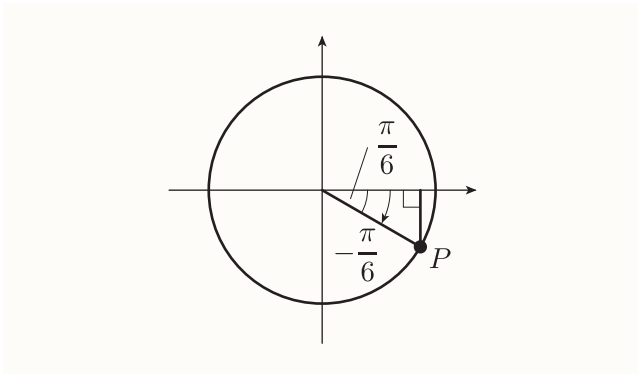
$$\sin(-3\pi) = \sin\pi = 0,$$

$$\cos(-3\pi) = \cos\pi = -1,$$

$$\tan(-3\pi) = \tan\pi = 0.$$

Solution to Exercise 8

(a)



The right-angled triangle in the diagram has acute angle $\pi/6$ at the origin. Hence the length of its horizontal side is

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

and the length of its vertical side is

$$\sin \frac{\pi}{6} = \frac{1}{2}.$$

So the coordinates of P are

$$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2} \right).$$

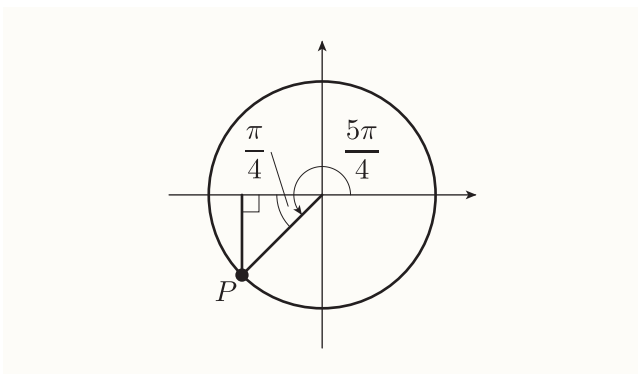
Hence

$$\sin \left(-\frac{\pi}{6} \right) = -\frac{1}{2},$$

$$\cos \left(-\frac{\pi}{6} \right) = \frac{\sqrt{3}}{2},$$

$$\tan \left(-\frac{\pi}{6} \right) = \left(-\frac{1}{2} \right) / \left(\frac{\sqrt{3}}{2} \right) = -\frac{1}{\sqrt{3}}.$$

(b)



The right-angled triangle in the diagram has acute angle $\pi/4$ at the origin. Hence the length of its horizontal side is

$$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

and the length of its vertical side is

$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

So the coordinates of P are

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right).$$

Hence

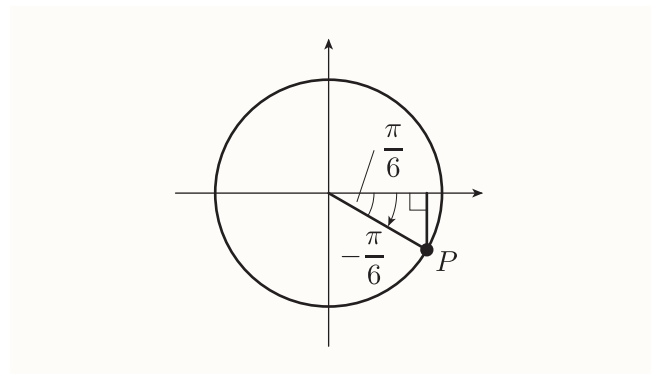
$$\sin \left(\frac{5\pi}{4} \right) = \frac{1}{\sqrt{2}},$$

$$\cos \left(\frac{5\pi}{4} \right) = -\frac{1}{\sqrt{2}},$$

$$\tan \left(\frac{5\pi}{4} \right) = \left(\frac{1}{\sqrt{2}} \right) / \left(-\frac{1}{\sqrt{2}} \right) = -1.$$

Solution to Exercise 9

(a) The answers to this part should match those to Exercise 8(a) above.



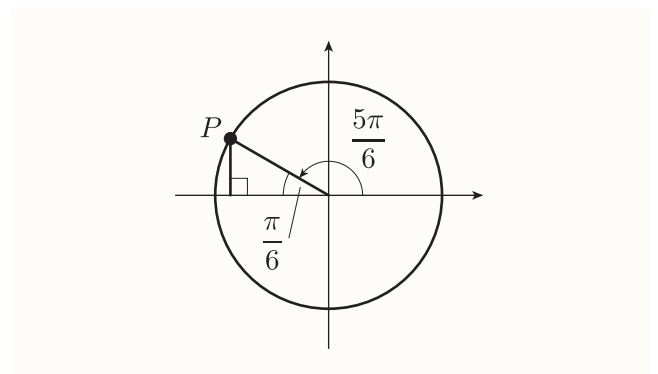
Since $-\pi/6$ is a fourth-quadrant angle, its sine and tangent are negative and its cosine is positive. Hence

$$\sin \left(-\frac{\pi}{6} \right) = -\sin \frac{\pi}{6} = -\frac{1}{2},$$

$$\cos \left(-\frac{\pi}{6} \right) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2},$$

$$\tan \left(-\frac{\pi}{6} \right) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

(b)



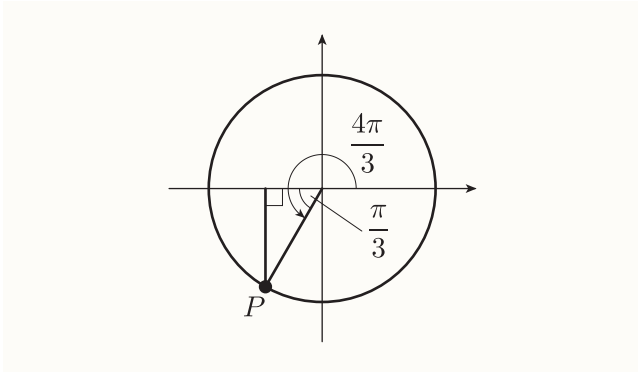
Since $5\pi/6$ is a second-quadrant angle, its cosine and tangent are negative and its sine is positive. Hence

$$\sin\left(\frac{5\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2},$$

$$\cos\left(\frac{5\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2},$$

$$\tan\left(\frac{5\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}.$$

(c)



Since $4\pi/3$ is a third-quadrant angle, its sine and cosine are negative and its tangent is positive. Hence

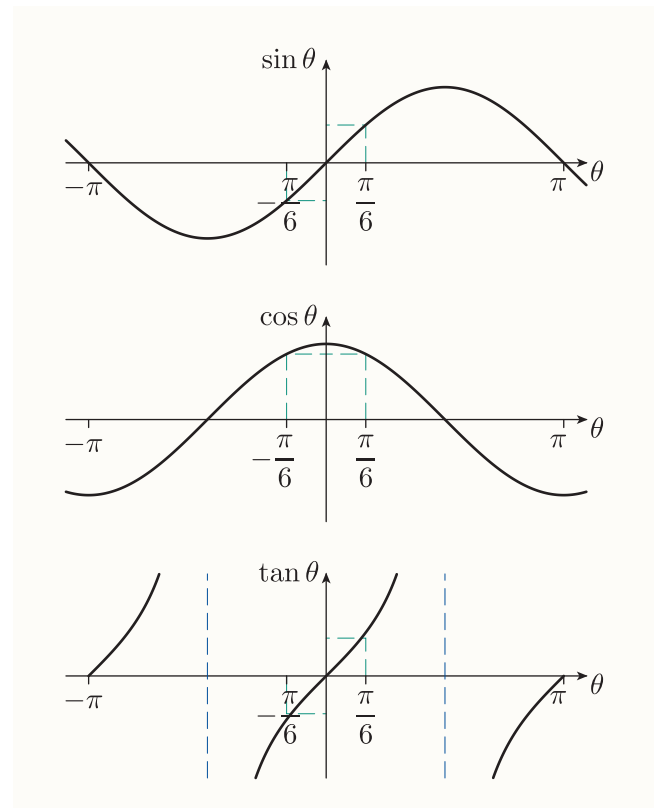
$$\sin\left(\frac{4\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2},$$

$$\cos\left(\frac{4\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2},$$

$$\tan\left(\frac{4\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}.$$

Solution to Exercise 10

(a)



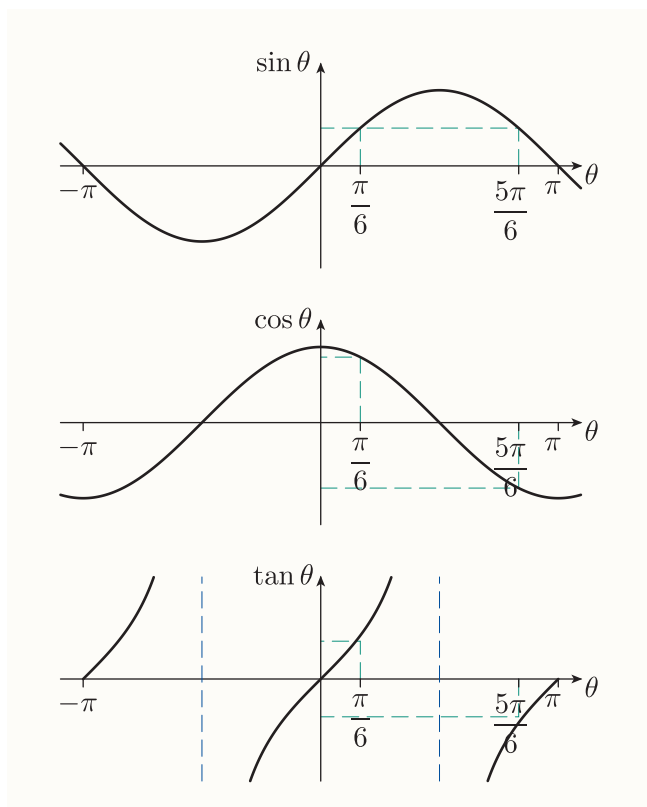
The symmetry of the graphs gives

$$\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2},$$

$$\cos\left(-\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},$$

$$\tan\left(-\frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}.$$

(b)



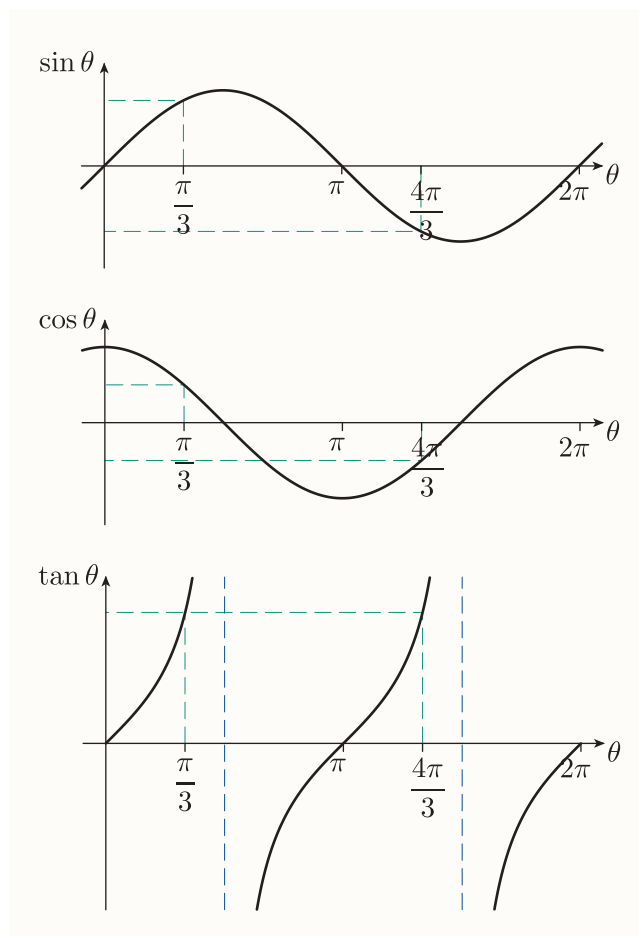
The symmetry of the graphs gives

$$\sin\left(\frac{5\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2},$$

$$\cos\left(\frac{5\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2},$$

$$\tan\left(\frac{5\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}.$$

(c)



The symmetry of the graphs gives

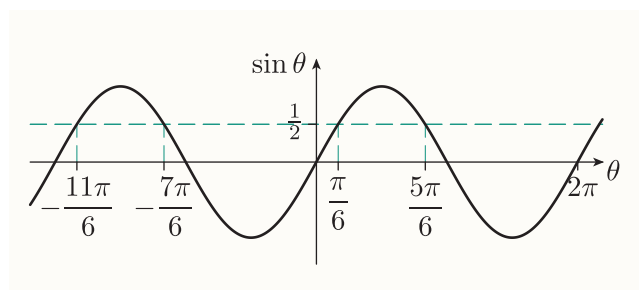
$$\sin\left(\frac{4\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2},$$

$$\cos\left(\frac{4\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2},$$

$$\tan\left(\frac{4\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}.$$

Solution to Exercise 11

(a)



One solution of the equation $\sin \theta = \frac{1}{2}$ is

$$\theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

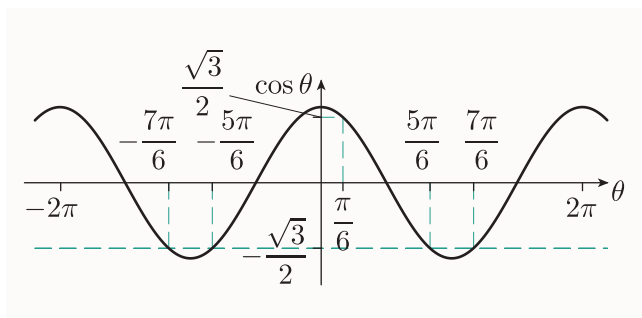
By the symmetry of the graph, for θ between -2π and 2π , the solutions are

$$\theta = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6} \text{ and } \frac{5\pi}{6}.$$

That is, the solutions between -360° and 360° are

$$\theta = -330^\circ, -210^\circ, 30^\circ \text{ and } 150^\circ.$$

(b)



A solution of the equation

$$\cos \theta = \frac{\sqrt{3}}{2}$$

is

$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$$

By the symmetry of the graph, for θ between -2π and 2π , the solutions of the equation

$$\cos \theta = -\frac{\sqrt{3}}{2}$$

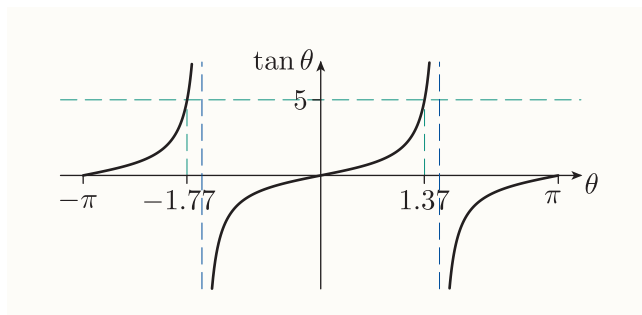
are

$$\theta = -\frac{7\pi}{6}, -\frac{5\pi}{6}, \frac{5\pi}{6} \text{ and } \frac{7\pi}{6}.$$

That is, the solutions between -360° and 360° are

$$\theta = -210^\circ, -150^\circ, 150^\circ \text{ and } 210^\circ.$$

Solution to Exercise 12



One solution of the equation $\tan \theta = 5$ is

$$\theta = \tan^{-1} 5 = 1.373\,400\dots$$

This solution is between $-\pi$ and π .

By the symmetry of the graph, there is a second solution between $-\pi$ and π , namely

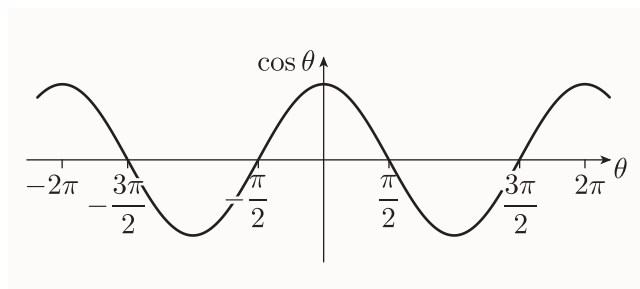
$$\theta = (\tan^{-1} 5) - \pi = -1.768\,191\dots$$

So, in the interval $-\pi$ to π , the solutions are

$$\theta = 1.37 \text{ and } -1.77 \text{ (to 2 d.p.)}.$$

Solution to Exercise 13

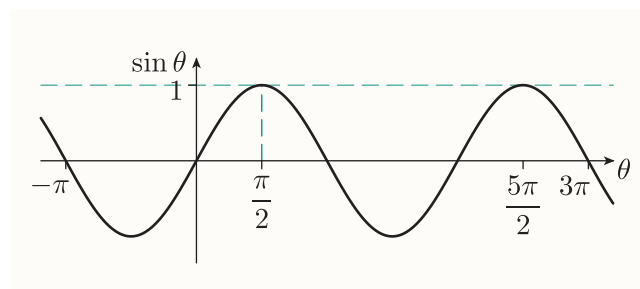
(a)



The graph shows that the solutions in the given interval are

$$-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2} \text{ and } \frac{3\pi}{2}.$$

(b)



The graph shows that the solutions in the given interval are

$$\frac{\pi}{2} \text{ and } \frac{5\pi}{2}.$$

Solution to Exercise 14

We have $b = 9$, $A = 24^\circ$ and $C = 53^\circ$. The third angle of the triangle is

$$B = 180^\circ - 24^\circ - 53^\circ = 103^\circ.$$

By the sine rule, the remaining side lengths are

$$a = \frac{b \sin A}{\sin B} = \frac{9 \sin 24^\circ}{\sin 103^\circ} = 3.8 \text{ (to 1 d.p.)}$$

$$c = \frac{b \sin C}{\sin B} = \frac{9 \sin 53^\circ}{\sin 103^\circ} = 7.4 \text{ (to 1 d.p.)}.$$

Hence, in the triangle ABC , we have $a = 3.8$, $b = 9$, $c = 7.4$, $A = 24^\circ$, $B = 103^\circ$ and $C = 53^\circ$.

Solution to Exercise 15

The sine rule gives

$$\frac{\sin 75.4^\circ}{13.8} = \frac{\sin B}{6.5}.$$

So

$$\sin B = \frac{6.5 \times \sin 75.4}{13.8}.$$

Now

$$\sin^{-1}\left(\frac{6.5 \times \sin 75.4}{13.8}\right) = 27.1167\dots^\circ$$

so

$$B = 27.1167\dots^\circ$$

or

$$B = 180^\circ - 27.1167\dots^\circ = 152.8832\dots^\circ.$$

However $b < a$, so B must be smaller than A .

Hence $B = 27.1167\dots^\circ$. So

$$\begin{aligned} C &= 180^\circ - 75.4^\circ - 27.1167\dots^\circ \\ &= 77.48^\circ \text{ (to 2 d.p.)}. \end{aligned}$$

Solution to Exercise 16

(a) By the sine rule,

$$\begin{aligned} \sin B &= \frac{b \sin C}{c} \\ &= \frac{14 \sin 20^\circ}{7} \\ &= 0.6840\dots \end{aligned}$$

This gives the possible solutions

$$\begin{aligned} B &= \sin^{-1}(0.684040\dots) \\ &= 43.1601\dots^\circ \\ &= 43.2^\circ \text{ (to 1 d.p.)} \end{aligned}$$

and

$$B = 180^\circ - 43.2^\circ = 136.8^\circ \text{ (to 1 d.p.)}.$$

(b) Given that BC , of length a , is the longest side of the triangle, A must be the largest angle of the triangle. This rules out the possibility that $B = 136.8^\circ$ (to 1 d.p.), which would make B , rather than A , the largest angle. This means that $B = 43.2^\circ$ (to 1 d.p.), so that

$$\begin{aligned} A &= 180^\circ - 20^\circ - 43.1601\dots^\circ \\ &= 116.8398\dots^\circ \\ &= 116.8 \text{ (to 1 d.p.)}. \end{aligned}$$

The only further value required is that of a , which is given by

$$\begin{aligned} a &= \frac{c \sin A}{\sin C} \\ &= \frac{7 \sin 116.8398\dots^\circ}{\sin 20^\circ} \\ &= 18.3 \text{ (to 1 d.p.)}. \end{aligned}$$

Solution to Exercise 17

(a) Using the cosine rule in the form

$$c^2 = a^2 + b^2 - 2ab \cos C$$

gives

$$c^2 = 4.6^2 + 5.4^2 - 2 \times 4.6 \times 5.4 \cos 108^\circ.$$

This gives $c = \sqrt{65.67196\dots}$.

So $c = 8.10382\dots = 8.1038$ (to 4 d.p.).

(b) Since $C > 90^\circ$, both A and B must be less than 90° . The sine rule gives

$$\frac{\sin A}{4.6} = \frac{\sin B}{5.4} = \frac{\sin 108^\circ}{8.1038\dots}.$$

So

$$\sin A = \frac{4.6 \times \sin 108^\circ}{8.1038\dots}$$

which gives $A = 32.6735^\circ$ (to 4 d.p.), and

$$\sin B = \frac{5.4 \times \sin 108^\circ}{8.1038\dots}$$

which gives $B = 39.3265^\circ$ (to 4 d.p.).

(A quick check shows that the angles add up to 180° .)

Solution to Exercise 18

Applying the cosine rule in the form

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

gives

$$\cos A = \frac{14^2 + 11^2 - 6^2}{2 \times 14 \times 11} = 0.912337\dots,$$

so

$$\begin{aligned} A &= \cos^{-1}(0.912337\dots) \\ &= 24.1695\dots^\circ \\ &= 24.17^\circ \text{ (to 2 d.p.)}. \end{aligned}$$

Applying the cosine rule in the form

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

gives

$$\cos B = \frac{6^2 + 11^2 - 14^2}{2 \times 6 \times 11} = -0.295454\dots,$$

so

$$\begin{aligned} B &= \cos^{-1}(-0.295\,454\dots) \\ &= 107.1847\dots^\circ \\ &= 107.18^\circ \text{ (to 2 d.p.)} \end{aligned}$$

The remaining angle, C , could also be found by applying the cosine rule. However, it is quicker to note that

$$\begin{aligned} C &= 180^\circ - A - B \\ &= 180^\circ - 24.1695\dots^\circ - 107.1847\dots^\circ \\ &= 48.65^\circ \text{ (to 2 d.p.)} \end{aligned}$$

Solution to Exercise 19

A regular hexagon can be divided into exactly 6 equilateral triangles with sides of length 8 cm and each angle 60° ,

So the area of the hexagon is

$$\begin{aligned} &6 \times \frac{1}{2} \times 8 \times 8 \times \sin 60^\circ \\ &= 192 \sin 60^\circ \\ &= 96\sqrt{3} \\ &= 166.28 \text{ cm}^2 \text{ (to 2 d.p.)} \end{aligned}$$

Solution to Exercise 20

The equation is $y = mx + c$, where the gradient m is given by

$$m = \tan\left(\frac{3\pi}{4}\right) = -1.$$

So the equation is $y = -x + c$.

The coordinates $(5, 0)$ satisfy this equation, and substituting them in gives

$$0 = -5 + c, \quad \text{so} \quad c = 5.$$

The equation of the line is therefore

$$y = -x + 5.$$

Solution to Exercise 21

The identity

$$\sin^2 \theta + \cos^2 \theta = 1$$

can be rewritten as

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$$

So if $\cos \theta$ is positive, then

$$\cos \theta = +\sqrt{1 - \sin^2 \theta},$$

and if $\cos \theta$ is negative, then

$$\cos \theta = -\sqrt{1 - \sin^2 \theta}.$$

Solution to Exercise 22

$$(a) \sin\left(\frac{3\pi}{2}\right) = -1 \text{ and } \cos\left(\frac{3\pi}{2}\right) = 0, \text{ so}$$

$$\operatorname{cosec}\left(\frac{3\pi}{2}\right) = \frac{1}{\sin\left(\frac{3\pi}{2}\right)} = \frac{1}{-1} = -1,$$

$$\sec\left(\frac{3\pi}{2}\right) \text{ is not defined,}$$

$$\cot\left(\frac{3\pi}{2}\right) = \frac{\cos\left(\frac{3\pi}{2}\right)}{\sin\left(\frac{3\pi}{2}\right)} = \frac{0}{-1} = 0.$$

$$(b) \sin \pi = 0 \text{ and } \cos \pi = -1, \text{ so}$$

$$\operatorname{cosec} \pi \text{ is not defined,}$$

$$\sec \pi = \frac{1}{\cos \pi} = \frac{1}{-1} = -1,$$

$$\cot \pi \text{ is not defined.}$$

$$(c) \text{ Since } 3\pi/2 \text{ and } -\pi/2 \text{ differ by } 2\pi, \text{ the answers to this part are the same as the answers to part (a). That is,}$$

$$\operatorname{cosec}\left(-\frac{\pi}{2}\right) = -1,$$

$$\sec\left(-\frac{\pi}{2}\right) \text{ is not defined,}$$

$$\cot\left(-\frac{\pi}{2}\right) = 0.$$

Solution to Exercise 23

$$(a) \text{ By the angle difference identity for sine,}$$

$$\sin(x - \pi) = \sin x \cos \pi - \cos x \sin \pi.$$

$$\text{Also } \cos \pi = -1 \text{ and } \sin \pi = 0. \text{ So}$$

$$\sin(x - \pi) = -\sin x.$$

$$(b) \text{ By the angle sum identity for sine,}$$

$$\sin\left(x + \frac{\pi}{2}\right) = \sin x \cos \frac{\pi}{2} + \cos x \sin \frac{\pi}{2}.$$

$$\text{Also } \cos \frac{\pi}{2} = 0 \text{ and } \sin \frac{\pi}{2} = 1. \text{ So}$$

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x.$$

$$(c) \text{ By the angle sum identity for cosine,}$$

$$\cos(x + \pi) = \cos x \cos \pi - \sin x \sin \pi.$$

$$\text{Also } \cos \pi = -1 \text{ and } \sin \pi = 0. \text{ So}$$

$$\cos(x + \pi) = -\cos x.$$

(d) By the angle sum identity for tangent,

$$\tan(x + \pi) = \frac{\tan x + \tan \pi}{1 - \tan x \times \tan \pi}.$$

But $\tan \pi = 0$. So

$$\tan(x + \pi) = \frac{\tan x}{1} = \tan x.$$

Solution to Exercise 24

(a) Let $A = \frac{\pi}{6}$ and $B = \frac{\pi}{4}$. Then

$$A + B = \left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \frac{5\pi}{12}.$$

Also

$$\sin \frac{\pi}{6} = \frac{1}{2}, \quad \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

$$\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}, \quad \tan \frac{\pi}{4} = 1.$$

So

$$\begin{aligned} \sin \left(\frac{5\pi}{12}\right) &= \sin \frac{\pi}{6} \cos \frac{\pi}{4} + \cos \frac{\pi}{6} \sin \frac{\pi}{4} \\ &= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \\ &= \frac{1 + \sqrt{3}}{2\sqrt{2}} \\ &= \frac{1}{4} (\sqrt{2} + \sqrt{6}), \end{aligned}$$

$$\begin{aligned} \cos \left(\frac{5\pi}{12}\right) &= \cos \frac{\pi}{6} \cos \frac{\pi}{4} - \sin \frac{\pi}{6} \sin \frac{\pi}{4} \\ &= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{3} - 1}{2\sqrt{2}} \\ &= \frac{1}{4} (\sqrt{6} - \sqrt{2}), \end{aligned}$$

$$\begin{aligned} \tan \left(\frac{5\pi}{12}\right) &= \frac{\tan \frac{\pi}{6} + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{6} \tan \frac{\pi}{4}} \\ &= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}} \times 1} \\ &= \frac{1 + \sqrt{3}}{\sqrt{3} - 1} \\ &= \frac{(1 + \sqrt{3})(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)} \\ &= \frac{1 + 2\sqrt{3} + 3}{3 - 1} \\ &= \frac{4 + 2\sqrt{3}}{2} \\ &= 2 + \sqrt{3}. \end{aligned}$$

(b) We have

$$\frac{5\pi}{4} = \pi + \frac{\pi}{4}.$$

Also

$$\sin(\pi) = 0, \quad \cos(\pi) = -1, \quad \tan(\pi) = 0$$

and

$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \tan \frac{\pi}{4} = 1.$$

So

$$\begin{aligned} \sin \left(\frac{5\pi}{4}\right) &= \sin \pi \cos \frac{\pi}{4} + \cos \pi \sin \frac{\pi}{4} \\ &= 0 - 1 \times \frac{1}{\sqrt{2}} \\ &= -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}, \end{aligned}$$

$$\begin{aligned} \cos \left(\frac{5\pi}{4}\right) &= \cos \pi \cos \frac{\pi}{4} - \sin \pi \sin \frac{\pi}{4} \\ &= -1 \times \frac{1}{\sqrt{2}} - 0 \\ &= -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}, \end{aligned}$$

$$\begin{aligned} \tan \left(\frac{5\pi}{4}\right) &= \frac{\tan \pi + \tan \frac{\pi}{4}}{1 - \tan \pi \tan \frac{\pi}{4}} \\ &= \frac{0 + 1}{1 - 0} = 1. \end{aligned}$$

(Alternatively, since

$$\tan \theta = \frac{\sin \theta}{\cos \theta},$$

we have

$$\begin{aligned}\tan\left(\frac{5\pi}{4}\right) &= \frac{\sin\left(\frac{5\pi}{4}\right)}{\cos\left(\frac{5\pi}{4}\right)} \\ &= \left(-\frac{1}{\sqrt{2}}\right) / \left(-\frac{1}{\sqrt{2}}\right) = 1.\end{aligned}$$

Solution to Exercise 25

(a) By a half-angle identity,

$$\begin{aligned}\sin^2\left(\frac{5\pi}{8}\right) &= \frac{1}{2} \left(1 - \cos\left(\frac{5\pi}{4}\right)\right) \\ &= \frac{1}{2} \left(1 - \left(-\frac{\sqrt{2}}{2}\right)\right) \\ &= \frac{1}{2} \left(1 + \frac{\sqrt{2}}{2}\right) \\ &= \frac{1}{4} (2 + \sqrt{2}).\end{aligned}$$

Now $0 < \frac{5\pi}{8} < \pi$, so $\sin\left(\frac{5\pi}{8}\right) > 0$.

Hence

$$\begin{aligned}\sin\left(\frac{5\pi}{8}\right) &= \sqrt{\frac{1}{4} (2 + \sqrt{2})} \\ &= \frac{1}{2} \sqrt{2 + \sqrt{2}}.\end{aligned}$$

(b) By a half-angle identity,

$$\begin{aligned}\cos^2\left(\frac{5\pi}{8}\right) &= \frac{1}{2} \left(1 + \cos\left(\frac{5\pi}{4}\right)\right) \\ &= \frac{1}{2} \left(1 - \frac{\sqrt{2}}{2}\right) \\ &= \frac{1}{4} (2 - \sqrt{2}).\end{aligned}$$

Now $\frac{\pi}{2} < \frac{5\pi}{8} < \pi$, so $\cos\left(\frac{5\pi}{8}\right) < 0$.

Hence

$$\begin{aligned}\cos\left(\frac{5\pi}{8}\right) &= -\sqrt{\frac{1}{4} (2 - \sqrt{2})} \\ &= -\frac{1}{2} \sqrt{2 - \sqrt{2}}.\end{aligned}$$

(c) By a half-angle identity,

$$\begin{aligned}\sin^2\left(\frac{5\pi}{24}\right) &= \frac{1}{2} \left(1 - \cos\left(\frac{5\pi}{12}\right)\right) \\ &= \frac{1}{2} \left(1 - \frac{1}{4} (\sqrt{6} - \sqrt{2})\right) \\ &= \frac{1}{2} \left(\frac{1}{4} (4 - \sqrt{6} + \sqrt{2})\right) \\ &= \frac{1}{8} (4 - \sqrt{6} + \sqrt{2}).\end{aligned}$$

Now $0 < \frac{5\pi}{24} < \frac{\pi}{2}$, so $\sin\left(\frac{5\pi}{24}\right) > 0$.

Hence

$$\begin{aligned}\sin\left(\frac{5\pi}{24}\right) &= \sqrt{\frac{1}{8} (4 - \sqrt{6} + \sqrt{2})} \\ &= \sqrt{\frac{1}{16} (8 - 2\sqrt{6} + 2\sqrt{2})} \\ &= \frac{1}{4} \sqrt{8 - 2\sqrt{6} + 2\sqrt{2}}.\end{aligned}$$

(d) By a half-angle identity,

$$\begin{aligned}\cos^2\left(\frac{5\pi}{24}\right) &= \frac{1}{2} \left(1 + \cos\left(\frac{5\pi}{12}\right)\right) \\ &= \frac{1}{2} \left(1 + \frac{1}{4} (\sqrt{6} - \sqrt{2})\right) \\ &= \frac{1}{8} (4 + \sqrt{6} - \sqrt{2}).\end{aligned}$$

Now $0 < \frac{5\pi}{24} < \frac{\pi}{2}$, so $\cos\left(\frac{5\pi}{24}\right) > 0$.

Hence

$$\begin{aligned}\cos\left(\frac{5\pi}{24}\right) &= \sqrt{\frac{1}{8} (4 + \sqrt{6} - \sqrt{2})} \\ &= \sqrt{\frac{1}{16} (8 + 2\sqrt{6} - 2\sqrt{2})} \\ &= \frac{1}{4} \sqrt{8 + 2\sqrt{6} - 2\sqrt{2}}.\end{aligned}$$